ISRAEL JOURNAL OF MATHEMATICS, Vol. 67, No. 1, 1989

A NOTE ON TILDE ALGEBRAS

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ABSTRACT

A closed set E is constructed so that $A^{\sim}(E)$ is an inseparable Banach space but its maximal ideal space is E.

Introduction

To each closed set E in the circle T we attach the algebras $A(E)$, $A^{\sim}(E)$, and $A^*(E)$. For examples, history, and background, we refer to [5] and [2, Ch. 12]. The algebra $A(E)$ is the set of restrictions to E of absolutely convergent Fourier series, i.e. of the so-called Wiener algebra, provided with the usual infimum as a norm. Next, $A^{\sim}(E)$ — the tilde algebra — is defined so that its unit ball is the closure, in $C(E)$, of the unit ball of $A(E)$. Alternatively, an element f of $C(E)$ belongs to $A^{\sim}(E)$ and has norm $\leqq c$ in $A^{\sim}(E)$ if

$$\left|\int f d\mu\right| \leqq c \, \| \mu \|_{PM} \qquad \text{for all } \mu \text{ in } M(E);$$

$\| \mu \|_{PM}$ is the supremum of $\hat{\mu}$. Finally, an element g of $C(E)$ belongs to $A^*(E)$ and has norm $\leqq c$ in $A^*(E)$ if

$$\left|\int g d\mu\right| \leqq c \limsup |\hat{\mu}(k)| \qquad \text{for all } \mu \text{ in } M(E).$$

The importance of $A^*(E)$ is seen from the (symbolic) formula $A^{\sim}(E) = A(E) + A^*(E)$. In [5] a set E is constructed so that $A(E)$ is a proper, dense subspace of $A^{\sim}(E)$ (see also [2, pp. 394–401]). That is indeed a *tour de force* and we obtain the example in the Abstract with little more than a variation on

† Supported in part by the National Science Foundation.

Received April 1, 1987 and in revised form October 9, 1988

the architecture of [5] and estimation of Fourier coefficients. Little is needed from the deeper theory of the algebra A.

1. E will be a disjoint union $\bigcup_2^\infty E_r \cup \{0\}$, each E_r being a Kronecker set (so the sets E_r shrink to 0). The sets E_r will satisfy a separation condition $\operatorname{diam} E_r \leqq \frac{1}{2} d(E_r, E\setminus E_r)$ so the partition functions w_r are available ($w_r = 1$ on $E\setminus E_r$, $w_r = 0$ on $E\setminus E_r$ and $\| w_r \|_A \leqq 3$).

Along with the sets E_r, we construct special probability measures μ_r, whose supports are exactly the sets E_r, with the following property. For every choice of integers $2 \leqq r_1 < r_2 < \cdots < r_n$ and every trigonometric polynomial p, such that $\int | p | d\mu_{r_j} < 1$ for each j, we have, setting $\lambda_j = p \cdot \mu_{r_j}$,

$$\limsup \sum_2^N |\hat{\lambda}_j(k)| \leqq cN^{2/3} \quad \text{as } |k| \to +\infty, \quad \text{with } c = c(E) < +\infty. \tag{1}$$

Now suppose g has norm < 1 in $A^*(E)$. We'll prove that the numbers $a_r \equiv \| g \|_{C(E_r)}$ satisfy $\Sigma_2^\infty a_r^4 \leqq c'$. Clearly $a_r \leqq 1$, and for each choice of $r_1, r_2, \ldots, r_N$ there is a polynomial such that $\int | p | d\mu_{r_j} < 1$ for each j, and $2 \operatorname{Re} \int g \cdot p d\mu_{r_j} \geqq a_{r_j}$. Then

$$\tfrac{1}{2} \sum a_{r_j} \leqq \left\langle g, \sum p d\mu_{r_j} \right\rangle \leqq cN^{2/3}$$

by the definition of $A^*(E)$ and inequality (1). Considering the decreasing re-arrangement of the sequence $(a_r)_1^\infty$, we get $\Sigma\, a_r^4 \leqq (4C)^4$. In each set E_r, g coincides with an element f_r of $A(T)$, having norm at most $2a_r$. The two series $\Sigma\, w_r f_r^4$ and $\Sigma\, w_r f_r^5$ converge absolutely and represent g^4 and g^5 in E (since $g(0) = 0$).

Let ϕ be a complex homorphism of $A^\sim(E)$, whence $\phi(f) \equiv f(t_0)$ for some t_0 in E and all f in $A(E)$. For each g in $A^*(E)$ we have g^4, g^5 in $A(E)$, whence $\phi(g) = g(t_0)$; since $A^\sim(E) = A(E) + A^*(E)$ we have $\phi(g) \equiv g(t_0)$ identically; that is, the maximal ideal space of $A^\sim(E)$ is E itself. (A similar device occurs in [1] for tensor algebras and tilde tensor algebras.)

2. The set E

We denote the interval $[2^{-\nu}, 2^{-\nu} + 2^{-\nu-3}]$ by I_ν ($\nu = 1, 2, 3, \ldots$) and inside each I_ν are draw 2^ν congruent intervals J_r ($2^\nu \leqq r < 2^{\nu+1}$) whose centers form an arithmetic progression. The intervals J_r are supposed to be so small that they have the separation property mentioned before. Let $\mu_{r,0}$ be the normalized

Lebesgue measure in J_r. Using the Rudin–Shapiro polynomials ([2, p. 33], [4, p. 33], [3, p. 34]) we find a sequence $\varepsilon_r = \pm 1$ with this property: for the measure $\lambda_\nu = \Sigma_{2^\nu}^{2^{\nu+1}-1} \varepsilon_r \mu_{r,0}$ we have $|\hat{\lambda}_\nu| \leqq 10 \cdot 2^{\nu/2}$. (As will be clear later, our proof uses much less than this.) Let h_ν be the function equal to $2^{-\nu/3}\varepsilon_r$ on the intervals J_r in I_ν (i.e. $2^\nu \leqq r < 2^{\nu+1}$), and null on the remaining intervals.

We shall construct $E_r \subseteq J_r$ so that (besides the properties already explained) each sum $\Sigma\, a_\nu h_\nu$, with $\sup |a_\nu| = 1$, belongs to $A^{\sim}(E)$, and has norm between two positive constants $a_1(E)$, $a_2(E)$. This yields at once the inseparability of the Banach space $A^{\sim}(E)$. In particular each h_ν has norm $\geqq a_1 > 0$ in $A^{\sim}(E)$; this is the most difficult inequality we shall encounter. The estimations used in the sequence of approximations are collected in the next section.

3. Inequalities on Fourier coefficients

LEMMA 1. *Let* $-1 \leqq b \leqq 1$ *and let* $L = 1, 2, 3, \ldots$. *Then there is a smooth function* $u \geqq 0$ *on* T, *such that* $\hat{u}(0) = 1$, $|\hat{u}(k)| \leqq \frac{1}{2}|b| + L^{-1/3}$ *for* $k \neq 0$, *vanishing off the open set* W:

$$W: \left| L^{-1} \sum_1^L (\cos 3^m t) - \tfrac{1}{2} b \right| < L^{-1/3}.$$

PROOF. Let $v(t) = \Pi_1^L (1 + b \cos 3^m t)$ so that $v(t) \geqq 0$ and $v(t)dt/2\pi$ defines a probability measure σ on T. With respect to σ we have expected values

$$E(\cos 3^m t) = \tfrac{1}{2} b, \qquad 1 \leqq m \leqq L,$$

$$E(\cos 3^m t \cos 3^q t) = \tfrac{1}{4} b^2, \qquad 1 \leqq m < q \leqq L.$$

Thus the variables $\cos 3^m t - \frac{1}{2} b$ $(1 \leqq m \leqq L)$ are orthogonal, with mean 0, variance $< \frac{1}{2}$ (by calculation). Thus there is a function $u(t) \geqq 0$, vanishing off W, such that $\hat{u}(0) = 1$ and $\int |u(t) - v(t)|\, dt < 2\pi \cdot L^{-1/3}$; and since W is open, u can be chosen to be smooth. Now $|\hat{v}(k)| \leqq \frac{1}{2}|b|$ for $k \neq 0$, whence $|\hat{u}(k)| < \frac{1}{2}|b| + L^{-1/3}$.

LEMMA 2. *Let* v *be periodic with an absolutely convergent Fourier series, let* ϕ *be a real function in* $C(T)$ *and* f *belong to* $L^1(T)$ *with norm* $\int |f(t)|/2\pi$. *Let* $c_{k,N}$ *denote the Fourier coefficients of* $v(Nt - \phi(t))f(t)$ *for* $N = 1, 2, 3, \ldots$. *Then*

(i) $|c_{k,N}| \leqq \|f\|_1 \cdot \|v\|_{PM} + \varepsilon_N$ *where* $\varepsilon_N \to 0$ *as* $N \to +\infty$, *and is independent of* k. *Moreover, if* $\hat{v}(0) = 0$, *then*

(ii) $|c_{k,N}| \leqq \varepsilon_N$ *for* $|k| \leqq N/2$, *and* ε_N *has the same meaning as before.*

PROOF. This is a classical procedure. By hypothesis $v(\theta) = \Sigma\, \alpha_n e(n\theta)$, with $e(\theta) \equiv e^{2\pi i\theta}$ and $\Sigma\, |\alpha_n| < +\infty$, and $\| v \|_{PM} = \max |\alpha_n|$. Thus

$$v(Nt - \phi(t))\, f(t) = \sum \alpha_n e(nNt - n\phi(t))\, f(t),$$

$$c_{k,N} = \sum_n \alpha_n \frac{1}{2\pi} \int e(nNt - kt) e(-n\phi(t))\, f(t) dt.$$

Each term in the sum has modulus at most $\| v \|_{PM} \cdot \| f \|_1$. For each k and N there is at most one integer n such that $|nN - k| < N/3$. The integral containing this n (if there is one) is estimated by $\| v \|_{PM} \cdot \| f \|_1$; the remaining contributions have a sum at most ε_N by the Riemann–Lebesgue Lemma and the convergence of $\Sigma\, |\alpha_n|$. This proves (i), and for the proof of (ii) we observe that when $|k| \leqq N/2$ and $|nN - k| < N/3$ then $n = 0$. The distinguished term in the previous inequality is therefore 0 (or it may be entirely absent) whence $|c_{k,N}| \leqq \varepsilon_N$.

4. Conclusion

The measures $\mu_{r,0}$ are now successively replaced by measures $\mu_{r,q}$ $(q = 1, 2, 3, \ldots)$ such that $\mu_r = \text{w*-lim}\, \mu_{r,q}$, and then E_r is the closed support of μ_r. When q is even, the operations are designed to make E_r a Kronecker set at the conclusion, so we operate only on one of the measures $\mu_{r,q}$, leaving all the others unchanged; we multiply one of the measures $\mu_{r,q}$ by $u_q(N_q t - \phi_q(t))$, where u_q is a smooth "peak-function" on T, ϕ_q is continuous, and N_q is a positive integer chosen by Lemma 2 and the prescriptions below. When q is odd we modify a large number of the measures $\mu_{r,q}$ so as to get estimates on finite sums $\Sigma\, a_v h_v$, with $a_v = 0$ or $a_v = \frac{1}{2}$. There will be an infinite sequence of operations to be arranged in a single sequence, but this offers no serious problem. It will be convenient to suppose that $\mu_{r,q} = \mu_{r,q+1}$ whenever $r \geqq 2^{q+1}$. Before the q-th step (resulting in measures $\mu_{r,q+1}$) is performed, we define a finite set $S(q)$ of integers, those satisfying one of two conditions:

(a) $|k| \leqq q$, or

(b) To define this we enumerate a dense sequence $(p_j)_1^\infty$ of trigonometric polynomials in the space $C(T)$, taking $p_1 \equiv 1$. We add to $S(q)$ all integers satisfying one or more of the inequalities

$$\left| \int p_j(t) e(-kt) d\mu_{r,q}(t) \right| > 4^{-q-2}$$

for some j and r in the range $1 \leqq j \leqq q$, $2 \leqq r \leqq 2^{q+1}$.

We can now say a bit more about the approximation process. When q is even, we operate on one of the measures $\mu_{r,q}$, replacing it by $u(Nt - \phi(t))\mu_{r,q}$, with $u \geqq 0$, $\hat{u}(0) = 1$, etc. This can be done with the aid of Lemma 2, taking $v = u - 1$; we can attain the following inequalities for $k \in S(q)$, $1 \leqq j \leqq q$, $2 \leqq r \leqq 2^{q+1}$:

$$|(p_j \cdot \mu_{r,q})^\wedge(k) - (p_j \cdot \mu_{r,q+1})^\wedge(k)| < 4^{-q-2}.$$

Strictly speaking $\mu_{r,q+1}$ must be adjusted to have total mass 1, but this can be controlled by the zeroth coefficient and $p_1 \equiv 1$.

For odd numbers q, the steps are more subtle. There is given a function $h = \Sigma\, a_\nu h_\nu$, with $a_\nu = 0$ or $\frac{1}{2}$ and we intend to approximate it on certain subsets of $I_\nu\,(1 \leqq \nu \leqq q)$ with a single function $L^{-1}\,\Sigma_1^L \cos 3^m Nt$. On $J_r\,(2^\nu \leqq r < 2^{\nu+1})$ the value of h is $a_\nu 2^{-\nu/3}\varepsilon_r$, and we use Lemma 1 with $b_r = 2a_\nu 2^{-\nu/3}\varepsilon_r$, $|b_r| < 1$. Lemma 1 gives a different function u for each b_r, but the approximating sums $L^{-1}\,\Sigma_1^L \cos 3^m Nt$ are the same. We take $L \geqq 2^q$, and choose N so large (by Lemma 2) that $|\hat{\mu}_{r,q} - \hat{\mu}_{r,q+1}| < 2 \cdot 2^{-\nu/3}$ for all values of k, on the range $2^\nu \leqq r < 2^{\nu+1}$, $1 \leqq \nu \leqq q$. For $1 \leqq j \leqq q$, we require the Fourier coefficients of $p_j\mu_{r,q} - p_j\mu_{r,q+1}$ to have absolute value at most $4 \cdot 2^{-\nu/3} \int |p_j(t)|\, d\mu_{r,q}$. (These estimates, for all values of k, are possible by Lemma 2.) We impose the same conditions, for $k \in S(q)$, as we did for even values of q. After adjusting the resulting measures $\mu_{r,q+1}$ to be probabilities we've completed the qth step. We check the necessary inequalities, beginning with the most difficult.

Let $\sigma_{\nu,q} = \Sigma'\, \varepsilon_r \mu_{r,q}$ (a sum over $2^\nu \leqq r < 2^{\nu+1}$). We claim that $|\hat{\sigma}_{\nu,q}| \leqq C 2^{2\nu/3}$ for all ν and q, and some constant C. This is certainly true for $q = 0$ with a constant C', as this is a very weak form of the Rudin–Shapiro inequalities. For each fixed k, the value $\hat{\sigma}_{\nu,q}(k)$ changes by at most 2 (when q is even) or $2 \cdot 2^{2\nu/3}$ (when q is odd). Thus, at the first instance q of the inequality $|\hat{\sigma}_{\nu,q}(k)| > (C' + 1)2^{2\nu/3}$, the upper bound $(C' + 3)2^{2\nu/3}$ remains valid. For some $r < 2^{q+1}$, we must also have $|\hat{\mu}_{r,q}(k)| > 2^{-\nu/3} \geqq 2^{-q/3}$ so that $k \in S(q)$. The construction shows that for all succeeding values $\tilde{q}$,

$$|\hat{\sigma}_{\nu,\tilde{q}}(k)| < (C' + 2)2^{2\nu/3} + 2^\nu \cdot 4^{-q-1} < (C' + 3)2^{2\nu/3}.$$

Now $\int h_\nu\, d\sigma_\nu = 2^\nu \cdot 2^{-\nu/3}$, since each μ_r is a probability measure, whence $\| h_\nu \| \geqq c > 0$ in the space $A^\sim(E)$. Thus each sum $\Sigma\, a_\nu h_\nu$ has norm at least $c \sup |a_\nu|$ in the space $A^\sim(E)$, since $\int h_\nu\, d\sigma_{\tilde{\nu}} = 0$ unless $\nu = \tilde{\nu}$. To prove the reverse inequality we can of course assume $a_\nu = 0$ or $a_\nu = \frac{1}{2}$. Let μ be a measure in E, and let μ_1 be that part of μ concentrated in $E \setminus \{0\}$. It is well known that

$\| \mu_1 \|_{PM} \leqq 2 \| \mu \|_{PM}$, and since $h(0) = 0$, we have $\int h d\mu = \int h d\mu_1$. Approximating μ_1 by measures with compact support in $E \setminus \{0\}$, our construction yields

$$\left| \int h d\mu \right| = \left| \int h d\mu_1 \right| \leqq \| \mu_1 \|_{PM} \leqq 2 \| \mu \|_{PM}.$$

Since $| h_\nu | \leqq 2^{\nu/3}$, we can pass from finite sums $\Sigma\, a_\nu h_\nu$ to infinite sums.

As for inequality (1), it will be sufficient to prove it for polynomials p_n in the sequence mentioned in (b). If $\int | p_n | d\mu_{r_j} < 1$ for $1 < r_1 < \cdots < r_N$, then the same inequality will hold with $\mu_{r_j,q}$ in place of μ_{r_j}, when $q > q^*$, say. If $q > q^*$ is so large that $2^q > r_N$ and $q > n$ then the Fourier coefficients of $p_n \cdot \mu_{r_j,q+1} - p_n \cdot \mu_{r_j,q}$ have modulus at most 2 when q is even and in that case all but one of the differences are 0; when q is odd the Fourier coefficient has modulus at most $4r_j^{-1/3}$ and then we note that

$$4 \sum_1^N r_j^{-1/3} \leqq 4N^{2/3}.$$

Suppose that k is an integer such that $\Sigma_1^N | \hat{\lambda}_j(k) | > 4N^{2/3}$ and that q^{**} is large enough to insure that all restrictions placed on q are effective. Since all the measures are absolutely continuous, there is a k^{**} so that

$$\left| \int p_n(t) \cdot e(-kt) d\mu^{**}_{r_j,q}(t) \right| < r_N^{-1} \qquad \text{for all } k > k^{**} \quad \text{or } k < -k^*.$$

If $| k | > k^{**}$, then there will be a first $\tilde{q} > q^{**}$ at which one of these inequalities is reversed, for some r_j. For this choice of k and $\tilde{q}$, we get $k \in S(\tilde{q})$. Passing to $q = +\infty$, each integral changes by at most 4^{1-q}, whence the sum for $j = 1, 2, 3, \ldots, N$ changes by at most $N \cdot 4^{1-q} < N^{-1/2}$. The sum at $\tilde{q}$, however, was at most $5N^{2/3}$; thus for $| k | > k^{**}$, $\Sigma_1^N | \hat{\lambda}_j(k) | < 6N^{2/3}$.

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